

CONFORMAL GEOMETRY AND FULLY NONLINEAR EQUATIONS

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To the memory of Professor S.S. Chern

ABSTRACT. This article is a survey of results involving conformal deformation of Riemannian metrics and fully nonlinear equations.

1. THE YAMABE EQUATION

One of the most important problems in conformal geometry is the Yamabe Problem, which is to determine whether there exists a conformal metric with constant scalar curvature on any closed Riemannian manifold. In what follows, let (M, g) be a Riemannian manifold, and let R denote the scalar curvature of g . Writing a conformal metric as $\tilde{g} = v^{\frac{4}{n-2}}g$, the Yamabe equation takes the form

$$(1.1) \quad 4\frac{n-1}{n-2}\Delta v + R \cdot v = \lambda \cdot v^{\frac{n+2}{n-2}},$$

where λ is a constant. These are the Euler-Lagrange equations of the *Yamabe functional*,

$$(1.2) \quad \mathcal{Y}(\tilde{g}) = Vol(\tilde{g})^{-\frac{n-2}{n}} \int_M R_{\tilde{g}} dvol_{\tilde{g}},$$

for $\tilde{g} \in [g]$, where $[g]$ denotes the conformal class of g . An important related conformal invariant is the *Yamabe invariant* of the conformal class $[g]$:

$$(1.3) \quad Y([g]) \equiv \inf_{\tilde{g} \in [g]} \mathcal{Y}(\tilde{g}).$$

The Yamabe problem has been completely solved through the results of many mathematicians, over a period of approximately thirty years. Initially, Yamabe claimed to have a proof in [Yam60]. The basic strategy was to prove the existence of a minimizer of the Yamabe functional through a sub-critical regularization technique. Subsequently, an error was found by N. Trudinger, who then gave a solution with a smallness assumption on the Yamabe invariant [Tru68]. Later, Aubin showed that the problem is solvable provided that

$$(1.4) \quad Y([g]) < Y([g_{\text{round}}]),$$

where $[g_{\text{round}}]$ denotes the conformal class of the round metric on the n -sphere, and verified this inequality for $n \geq 6$ and g not locally conformally flat [Aub76b], [Aub76a],

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[Aub98]. Schoen solved the remaining cases [Sch84]. It is remarkable that Schoen employed the positive mass conjecture from general relativity to solve these remaining most difficult cases.

An important fact is that $SO(n+1,1)$, the group of conformal transformations of the n -sphere S^n with the round metric, is non-compact. Likewise, the space of solutions to the Yamabe equation in the conformal class of the round sphere is non-compact. However, if (M, g) is compact, and not conformally equivalent to the round sphere, then the group of conformal transformations is compact [LF71], [Oba72]. A natural question is then whether the space of all unit-volume solutions (not just minimizers) to (1.1) is compact on an arbitrary compact manifold, provided (M, g) is not conformally equivalent to S^n with the round metric. Schoen solved this in the locally conformally flat case [Sch91], and produced unpublished lecture notes outlining a solution in certain other cases. Many other partial solutions have appeared, see for example [Dru04], [LZ04], [LZ05], [Mar05], [Sch89]. Schoen has recently announced the complete solution of the compactness problem in joint work with Khuri and Marques, assuming that the positive mass theorem holds in higher dimensions. The positive mass theorem is known to hold in the locally conformally flat case in all dimensions [SY79a], and in the general case in dimensions $n \leq 7$ [SY81], [LP87], and in any dimension if the manifold is spin [Wit81], [PT82], [LP87].

2. A FULLY NONLINEAR YAMABE PROBLEM

The equation (1.1) is a semi-linear equation, meaning the non-linearities only appear in lower order terms – second derivatives appear in a linear fashion. One may investigate other types of conformal curvature equations, which brings one into the realm of fully nonlinear equations. We recall the Schouten tensor

$$(2.1) \quad A_g = \frac{1}{n-2} \left(Ric - \frac{R}{2(n-1)}g \right),$$

where Ric denotes the Ricci tensor. This tensor arises naturally in the decomposition of the full curvature tensor

$$(2.2) \quad Riem = Weyl + A \odot g,$$

where \odot denotes the Kulkarni-Nomizu product [Bes87]. This equation also serves to define the Weyl tensor, which is conformally invariant. Thus the behaviour of the full curvature tensor under a conformal change of metric is entirely determined by the Schouten tensor. Let F denote any symmetric function of the eigenvalues, which is homogeneous of degree one, and consider the equation

$$(2.3) \quad F(\tilde{g}^{-1}A_{\tilde{g}}) = \text{constant}.$$

Note that the \tilde{g}^{-1} factor is present since only the eigenvalues of an endomorphism are well-defined. If we write a conformal metric as $\tilde{g} = e^{-2u}g$, the Schouten tensor transforms as

$$(2.4) \quad A_{\tilde{g}} = \nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g + A_g.$$

Therefore, equation (2.3) is equivalent to

$$(2.5) \quad F \left(g^{-1} \left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g + A_g \right) \right) = \text{constant} \cdot e^{-2u}.$$

Let σ_k denote the k th elementary symmetric function of the eigenvalues

$$(2.6) \quad \sigma_k = \sum_{i_1 < \dots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}.$$

For the case of $F = \sigma_k^{1/k}$, the equation (2.3) has become known as the σ_k -Yamabe equation:

$$(2.7) \quad \sigma_k^{1/k}(\tilde{g}^{-1} A_{\tilde{g}}) = \text{constant}.$$

In the context of exterior differential systems, they arose in a different form in Bryant and Griffiths' research on conformally invariant Poincaré-Cartan forms [BGG03], and these systems were shown to correspond to the σ_k -Yamabe equation in [Via00a].

For $1 \leq k \leq n$, we define the cone (in \mathbb{R}^n)

$$(2.8) \quad \Gamma_k^+ = \{\sigma_k > 0\} \cap \{\sigma_{k-1} > 0\} \cap \dots \cap \{\sigma_1 > 0\}.$$

These are well-known as ellipticity cones for the σ_k equation, see [Gär59], [Ivo83], [CNS85]. We will say that a metric g is strictly k -admissible if the eigenvalues of $g^{-1} A_g$ lie in Γ_k^+ at every point $p \in M$. It is an important fact that if the metric \tilde{g} is k -admissible, then the linearization of (2.7) at \tilde{g} is elliptic. On a compact manifold, if the background metric g is k -admissible, then (2.7) is necessarily elliptic at *any* solution [Via02]. Thus, a k -admissibility assumption on the background metric is an ellipticity assumption, reminiscent of the k -convexity assumption on domains for the k -Hessian equation [CNS85].

3. VARIATIONAL CHARACTERIZATION

In [Via00a], it was shown that in several cases, the σ_k -Yamabe equation is variational. Let \mathcal{M}_1 denote the set of unit volume metrics in the conformal class $[g_0]$.

Theorem 3.1. (*Viaclovsky* [Via00a]) *If $k \neq n/2$ and $(N, [g_0])$ is locally conformally flat, a metric $g \in \mathcal{M}_1$ is a critical point of the functional*

$$\mathcal{F}_k : g \mapsto \int_N \sigma_k(g^{-1} A_g) dV_g$$

restricted to \mathcal{M}_1 if and only if

$$\sigma_k(g^{-1} A_g) = C_k$$

for some constant C_k . If N is not locally conformally flat, the statement is true for $k = 1$ and $k = 2$.

For $k = 1$, this is of course well-known, as \mathcal{F}_1 is the Hilbert functional, [Hil72], [Sch89]. For $k = n/2$, in [Via00a] it was shown that the integrand is the non-Weyl part of the Chern-Gauss-Bonnet integrand. Therefore, $\mathcal{F}_{n/2}$ is necessarily constant in the locally conformally flat case (when $n = 4$, this holds in general, this will be

discussed in detail in Section 6 below). Nevertheless, in this case the equation is still variational, but with a different functional. Fix a background metric h , write $g = e^{-2u}h$, and let

$$(3.1) \quad \mathcal{E}_{n/2}(g) = \int_M \int_0^1 \sigma_{n/2} \left(-t \nabla_h^2 u + t^2 \nabla_h u \otimes \nabla_h u - \frac{1}{2} t^2 |\nabla_h u|^2 g_0 + A(h) \right) u dt dV_h,$$

then for any differentiable path of smooth conformal metrics g_t ,

$$(3.2) \quad \frac{d}{dt} \mathcal{E}_{n/2}(g_t) = \int_M \sigma_{n/2}(A_{g_t}) u d\text{vol}_{g_t}.$$

This fact was demonstrated in [BV04], see also [CY03]. This is valid also for $n = 4$, [CY95]. Recently, Sheng-Trudinger-Wang have given conditions on when the more general F -Yamabe equation is variational, see [STW05].

A natural question is: what are the critical metrics of the \mathcal{F}_k functionals, when considering all possible metric variations, not just conformal variations? It is a well-known result that the critical points of \mathcal{F}_1 restricted to space of unit volume metrics are exactly the Einstein metrics [Sch89]. But for $k > 1$, the full Euler-Lagrange equations are manifestly fourth order equations in the metric. However, in dimension three we have the following

Theorem 3.2. (*Gursky-Viaclovsky* [GV01]) *Let M be compact and of dimension three. Then a metric g with $\mathcal{F}_2[g] \geq 0$ is critical for \mathcal{F}_2 restricted to the space of unit volume metrics if and only if g has constant sectional curvature.*

A similar theorem was proved by Hu-Li for $n \geq 5$, but with the rather stringent condition that the metric be locally conformally flat [HL04]. We mention that Labbi studied some curvature quantities defined by H. Weyl which are polynomial in the full curvature tensor, and proved some interesting variational formulas [Lab04].

4. LIOUVILLE THEOREMS

We next turn to the uniqueness question. In the negative curvature case, the linearization of (2.3) is invertible, so the uniqueness question is trivial. However, in the positive curvature case the uniqueness question is non-trivial. In [Via00a], [Via00b] the following was proved

Theorem 4.1. (*Viaclovsky* [Via00a]) *Suppose (N, g_0) is of unit volume and has constant sectional curvature $K > 0$. Then for any $k \in \{1, \dots, n\}$, g_0 is the unique unit volume solution in its conformal class of*

$$(4.1) \quad \sigma_k(g^{-1}A_g) = \text{constant},$$

unless N is isometric to S^n with the round metric. In this case we have an $(n+1)$ -parameter family of solutions which are the images of the standard metric under the conformal diffeomorphisms of S^n .

For $k = 1$, the constant scalar curvature case, the theorem holds just assuming N is Einstein. This is the well-known theorem of Obata [Oba72].

This theorem falls under the category of a Liouville-type theorem. We let δ_{ij} be the Kronecker delta symbol, and write the conformal factor as $\tilde{g} = u^{-2}g$. In stereographic coordinates, the equation (4.1) is written

$$(4.2) \quad \sigma_k \left(u \cdot \frac{\partial^2 u}{\partial x^i \partial x^j} - \frac{|\nabla u|^2}{2} \delta_{ij} \right) = \text{constant}.$$

This equation is conformally invariant: if $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a conformal transformation (i.e., $T \in SO(n+1, 1)$), and $u(x)$ is a solution of (4.2), then

$$(4.3) \quad v(x) = |J(x)|^{-1/n} u(Tx)$$

is also a solution, where J is the Jacobian of T , see [Via00b].

The uniqueness theorem can then be restated as a Liouville Theorem in \mathbb{R}^n :

Theorem 4.2. (*Viaclovsky [Via00b]*) *Let $u(x) \in C^\infty(\mathbb{R}^n)$ be a positive solution to*

$$(4.4) \quad \sigma_k \left(u \cdot \frac{\partial^2 u}{\partial x^i \partial x^j} - \frac{|\nabla u|^2}{2} \delta_{ij} \right) = \text{constant}$$

for some $k \in \{1, \dots, n\}$. Suppose that $v(y) = |y|^2 \cdot u(\frac{y^1}{|y|^2}, \dots, \frac{y^n}{|y|^2})$ is smooth and

$$\lim_{y \rightarrow 0} v(y) > 0.$$

Then

$$u = a|x|^2 + b_i x^i + c,$$

where a , b_i , and c are constants.

The proof in [Via00a], [Via00b] requires the stringent growth condition at infinity. For the scalar curvature equation, $k = 1$, this theorem was proved without *any* assumption at infinity in the important paper by Caffarelli-Gidas-Spruck [CGS89], using the moving planes technique.

This analagous theorem for $k \geq 2$ is now known to hold without any condition on the behaviour at infinity. For $k = 2$, important work was done in [CGY02a], and [CGY03b], proving the Liouville Theorem for $n = 4, 5$, and for $n \geq 6$ with a finite volume assumption. Maria del Mar González proved a Liouville Theorem for σ_k , $n > 2(k+1)$ with a finite volume assumption [Gon04b].

Yanyan Li and Aobing Li proved the following theorem for all k in the important paper [LL03].

Theorem 4.3. (*Li-Li [LL03]*) *Let $u(x) \in C^\infty(\mathbb{R}^n)$ be a positive solution to*

$$(4.5) \quad \sigma_k \left(u \cdot \frac{\partial^2 u}{\partial x^i \partial x^j} - \frac{|\nabla u|^2}{2} \delta_{ij} \right) = \text{constant} > 0$$

for some $k \in \{1, \dots, n\}$, satisfying

$$(4.6) \quad u \cdot \frac{\partial^2 u}{\partial x^i \partial x^j} - \frac{|\nabla u|^2}{2} \delta_{ij} \in \Gamma_k^+.$$

Then

$$u = a|x|^2 + b_i x^i + c,$$

where a , b_i , and c are constants.

Their method is based on the moving planes technique. Subsequently, their results have been generalized to much more general classes of symmetric functions F , see [LL05b], [Li03], [LL05a], [Li02].

These types of Liouville theorems can be used in deriving a priori estimates for solutions of (4.1), we will discuss this below.

5. LOCAL ESTIMATES

We consider the σ_k -Yamabe equation

$$(5.1) \quad \sigma_k^{1/k} \left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g + A_g \right) = f(x) e^{-2u},$$

with $f(x) \geq 0$. A remarkable property of (5.1) was discovered in [GW03b]. It turns out that *local* estimates are satisfied, a fact which does not hold in general fully nonlinear equations. We say that $u \in C^2$ is k -admissible if $A_u \in \bar{\Gamma}_k^+$. For equation (5.1), Guan and Wang prove

Theorem 5.1. (*Guan-Wang [GW03b]*) *Let $u \in C^3(M^n)$ be a k -admissible solution of (5.1) in $B(x_0, \rho)$, where $x_0 \in M^n$ and $\rho > 0$. Then there is a constant*

$$C_0 = C_0(k, n, \rho, \|g\|_{C^2(B(x_0, \rho))}, \|f\|_{C^1(B(x_0, \rho))}),$$

such that

$$(5.2) \quad |\nabla u|^2(x) \leq C_0(1 + e^{-2 \inf_{B(x_0, \rho)} u})$$

for all $x \in B(x_0, \rho/2)$.

Let $u \in C^4(M^n)$ be a k -admissible solution of (5.1) in $B(x_0, \rho)$, where $x_0 \in M^n$ and $\rho > 0$. Then there is a constant

$$C_0 = C_0(k, n, \rho, \|g\|_{C^3(B(x_0, \rho))}, \|f\|_{C^2(B(x_0, \rho))}),$$

such that

$$(5.3) \quad |\nabla^2 u|(x) + |\nabla u|^2(x) \leq C_0(1 + e^{-2 \inf_{B(x_0, \rho)} u})$$

for all $x \in B(x_0, \rho/2)$.

These local estimates for (5.1) generalize the global estimates which were first proved in [Via02]. Subsequently, these results have been extended to much more general classes of symmetric functions F , see [Che05], [GW04], [GLW04b], [LL03], [Li06], [Wan06]. Estimates for solutions of σ_2 in dimension four were proved in [Han04] using integral methods.

Equipped with second derivative estimates, one then uses the work of Evans and Krylov [Eva82], [Kry83] to obtain $C^{2, \alpha}$ estimates, that is, a Hölder estimate on second derivatives. This is crucial – the importance of that work in this theory can not be overstated.

Consider a symmetric function

$$(5.4) \quad F : \Gamma \subset \mathbb{R}^n \rightarrow \mathbb{R}$$

with $F \in C^\infty(\Gamma) \cap C^0(\overline{\Gamma})$, where $\Gamma \subset \mathbb{R}^n$ is an open, symmetric, convex cone, and impose the following conditions:

(i) F is symmetric, concave, and homogenous of degree one.

(ii) $F > 0$ in Γ , and $F = 0$ on $\partial\Gamma$.

(iii) F is *elliptic*: $F_{\lambda_i}(\lambda) > 0$ for each $1 \leq i \leq n$, $\lambda \in \Gamma$.

We mention that Szu-Yu Chen proved local C^2 -estimates (5.3) for this general class of symmetric functions F [Che05].

An immediate corollary of these local estimates is an ϵ -regularity result:

Theorem 5.2. (*Guan-Wang* [GW03b]) *There exist constants $\epsilon_0 > 0$ and $C = C(g, \epsilon_0)$ such that any solution $u \in C^2(B(x_0, \rho))$ of (5.1) with*

$$(5.5) \quad \int_{B(x_0, \rho)} e^{-nu} dvol_g \leq \epsilon_0,$$

satisfies

$$(5.6) \quad \inf_{B(x_0, \rho/2)} u \geq -C + \log \rho.$$

Consequently, there is a constant

$$C_2 = C_2(k, n, \mu, \epsilon_0, \|g\|_{C^3(B(x_0, \rho))}),$$

such that

$$(5.7) \quad |\nabla^2 u|(x) + |\nabla u|^2(x) \leq C_2 \rho^{-2}$$

for all $x \in B(x_0, \rho/4)$.

This type of estimate is crucial in understanding *bubbling*, a phenomenon which is unavoidable when studying conformally invariant problems. It shows that non-compactness of the space of solutions can arise only through volume concentration.

6. DIMENSION FOUR

In dimension four, an important conformal invariant is

$$(6.1) \quad \mathcal{F}_2([g]) \equiv 4 \int_M \sigma_2(A_g) dV_g = \int_M \left(-\frac{1}{2} |Ric_g|^2 + \frac{1}{6} R_g^2 \right) dvol_g.$$

By the Chern-Gauss-Bonnet formula ([Bes87]),

$$(6.2) \quad 8\pi^2 \chi(M) = \int_M |W_g|^2 dvol_g + \mathcal{F}_2([g]).$$

Thus, the conformal invariance of \mathcal{F}_2 follows from the well known (pointwise) conformal invariance of the Weyl tensor W_g (see [Eis97]).

One of the most interesting results in this area is

Theorem 6.1. (*Chang-Gursky-Yang [CGY02b]*) *Let (M, g) be a closed 4-dimensional Riemannian manifold with positive scalar curvature. If $\mathcal{F}_2([g]) > 0$, then there exists a conformal metric $\tilde{g} = e^{-2u}g$ with $R_{\tilde{g}} > 0$ and $\sigma_2(\tilde{g}^{-1}A_{\tilde{g}}^1) > 0$ pointwise. In particular, the Ricci curvature of \tilde{g} satisfies*

$$0 < 2\text{Ric}_{\tilde{g}} < R_{\tilde{g}}\tilde{g}.$$

By combining this with some work of Margerin [Mar98] on the Ricci flow, the authors obtained the following remarkable integral sphere-pinching theorem:

Theorem 6.2. (*Chang-Gursky-Yang [CGY03a]*) *Let (M^4, g) be a smooth, closed four-manifold for which*

- (i) *the Yamabe invariant $Y([g]) > 0$, and*
- (ii) *the Weyl curvature satisfies*

$$(6.3) \quad \int_{M^4} |W|^2 d\text{vol} < 16\pi^2 \chi(M^4).$$

Then M^4 is diffeomorphic to either S^4 or \mathbb{RP}^4 .

The proof of Theorem 6.1 in [CGY02b] involved regularization by a fourth-order equation and relied on some delicate integral estimates. Subsequently, a more direct proof was given in [GV03a]: define the tensor

$$(6.4) \quad A_g^t = \frac{1}{2} \left(\text{Ric}_g - \frac{t}{6} R_g g \right).$$

Theorem 6.3. (*Gursky-Viaclovsky [GV03a]*) *Let (M, g) be a closed 4-dimensional Riemannian manifold with positive scalar curvature. If*

$$(6.5) \quad \mathcal{F}_2([g]) + \frac{1}{6}(1-t_0)(2-t_0)(Y([g]))^2 > 0,$$

for some $t_0 \leq 1$, then there exists a conformal metric $\tilde{g} = e^{-2u}g$ with $R_{\tilde{g}} > 0$ and $\sigma_2(A_{\tilde{g}}^{t_0}) > 0$ pointwise. This implies the pointwise inequalities

$$(6.6) \quad (t_0 - 1)R_{\tilde{g}}\tilde{g} < 2\text{Ric}_{\tilde{g}} < (2 - t_0)R_{\tilde{g}}\tilde{g}.$$

The proof involves a deformation of the equation through a path of fully nonlinear equations, and an application of the local estimates of Guan-Wang. A similar technique was applied the σ_k equations in the locally conformally flat case by Guan-Lin-Wang [GLW04a].

As applications of Theorem 6.3, consider two different values of t_0 . When $t_0 = 1$, we obtain the aforementioned result in [CGY02b]. The second application is to the spectral properties of a conformally invariant differential operator known as the *Paneitz operator*. Let δ denote the L^2 -adjoint of the exterior derivative d ; then the Paneitz operator is defined by

$$(6.7) \quad P_g \phi = \Delta^2 \phi + \delta \left(\frac{2}{3} R_g g - 2\text{Ric}_g \right) d\phi.$$

The Paneitz operator is conformally invariant, in the sense that if $\tilde{g} = e^{-2u}g$, then

$$(6.8) \quad P_{\tilde{g}} = e^{4u}P_g.$$

Since the volume form of the conformal metric \tilde{g} is $dvol_{\tilde{g}} = e^{-4u}dvol_g$, an immediate consequence of (6.8) is the conformal invariance of the Dirichlet energy

$$\langle P_{\tilde{g}}\phi, \phi \rangle_{L^2(M, \tilde{g})} = \langle P_g\phi, \phi \rangle_{L^2(M, g)}.$$

In particular, positivity of the Paneitz operator is a conformally invariant property, and clearly the kernel is invariant as well.

To appreciate the geometric significance of the Paneitz operator, define the associated *Q-curvature*, introduced by Branson:

$$(6.9) \quad Q_g = -\frac{1}{12}\Delta R_g + 2\sigma_2(g^{-1}A_g^1).$$

Under a conformal change of metric $\tilde{g} = e^{-2u}g$, the *Q-curvature* transforms according to the equation

$$(6.10) \quad -Pu + 2Q_g = 2Q_{\tilde{g}}e^{-4u},$$

see, for example, [BO91]. Note that

$$(6.11) \quad \int_M Q_g dvol_g = \frac{1}{2}\mathcal{F}_2([g]),$$

so the integral of the *Q-curvature* is conformally invariant.

An application of Theorem 6.3 with $t_0 = 0$, yields

Theorem 6.4. (*Gursky-Viaclovsky* [GV03a]) *Let (M, g) be a closed 4-dimensional Riemannian manifold with positive scalar curvature. If*

$$(6.12) \quad \int Q_g dvol_g + \frac{1}{6}(Y[g])^2 > 0,$$

then the Paneitz operator is nonnegative, and $\text{Ker } P = \{\text{constants}\}$. Therefore, by the results in [CY95], there exists a conformal metric $\tilde{g} = e^{-2u}g$ with $Q_{\tilde{g}} = \text{constant}$.

This yields new examples of manifolds admitting constant *Q-curvature* metrics, see [GV03a].

Another interesting application of σ_2 in dimension 4 was found in [Wan05], which gives a lower estimate on eigenvalues of the Dirac operator in terms of \mathcal{F}_2 on a spin manifold.

7. PARABOLIC METHODS

We recall the *Yamabe flow*,

$$(7.1) \quad \frac{d}{dt}g = -(R_g - r_g)g,$$

where r_g denotes the mean value of the scalar curvature. This flow was introduced by Hamilton, who proved existence of the flow for all time and proved convergence in the case of negative scalar curvature. The case of positive scalar curvature however

is highly non-trivial. The locally conformally flat case was studied in [Cho92] and [Ye94]. Schwetlick and Struwe [SS03] proved convergence for $3 \leq n \leq 5$ provided an certain energy bound on the initial metric is satisfied. In the beautiful paper [Bre05], Simon Brendle proved convergence for $3 \leq n \leq 5$ for *any* initial data.

In the fully nonlinear case, the following flow was first proposed in [GW03a]:

$$(7.2) \quad \begin{cases} \frac{d}{dt}g &= -(\log \sigma_k(g) - \log r_k(g)) \cdot g, \\ g(0) &= g_0, \end{cases}$$

where $r_k(g)$ is given by

$$r_k(g) = \exp \left(\frac{1}{\text{vol}(g)} \int_M \log \sigma_k(g) \, d\text{vol}(g) \right).$$

If $g = e^{-2u} \cdot g_0$, then equation (7.2) can be written as the following fully nonlinear flow

$$(7.3) \quad \begin{cases} 2 \frac{du}{dt} &= \log \sigma_k \left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + A_{g_0} \right) + 2ku - \log r_k \\ u(0) &= u_0. \end{cases}$$

Guan and Wang settled the locally conformally flat case:

Theorem 7.1. (*Guan-Wang [GW03a]*) *Suppose (M, g_0) be a compact, connected and locally conformally flat manifold. Assume that $A_{g_0} \in \Gamma_k^+$ and smooth, then flow (7.2) exists for all time $0 < t < \infty$ and $g(t) \in C^\infty(M)$ for all t . For any positive integer l , there exists a constant C depending only on g_0, k, n (independent of t) such that*

$$(7.4) \quad \|g\|_{C^l(M)} \leq C,$$

where the norm is taken with respect to the background metric g_0 . Furthermore, there are a positive number β and a smooth metric $g_\infty \in \Gamma_k^+$ such that

$$(7.5) \quad \sigma_k(A_{g_\infty}) = \beta,$$

and

$$(7.6) \quad \lim_{t \rightarrow \infty} \|g(t) - g_\infty\|_{C^1(M)} = 0,$$

for all l .

Guan and Wang employed a log flow (rather than a gradient flow) due to a technical reason in obtaining C^2 estimates. This solved the σ_k -Yamabe problem in the locally conformally flat case. Independently, Yanyan Li and Aobing Li solved the locally conformally flat σ_k -Yamabe problem using elliptic methods [LL03].

In a subsequent paper [GW04], Guan-Wang employed the log-flow for the quotient equations to obtain some interesting inequalities. Define the scale invariant functionals

$$(7.7) \quad \mathcal{F}_k(g) = (\text{Vol}(g))^{-\frac{n-2k}{n}} \int_M \sigma_k(g^{-1} A_g) dV_g.$$

Theorem 7.2. (*Guan-Wang [GW04]*) Suppose that (M, g_0) is a compact, oriented and connected locally conformally flat manifold with $A_{g_0} \in \Gamma_k^+$ smooth and conformal metric g with $A_g \in \Gamma_k^+$. Let $0 \leq l < k \leq n$.

(A). Sobolev type inequality: If $0 \leq l < k < \frac{n}{2}$, then there is a positive constant $C_S = C_S([g_0], n, k, l)$ depending only on n, k, l and the conformal class $[g_0]$ such that

$$(7.8) \quad (\mathcal{F}_k(g))^{\frac{1}{n-2k}} \geq C_S (\mathcal{F}_l(g))^{\frac{1}{n-2l}}.$$

(B). Conformal quermassintegral type inequality: If $n/2 \leq k \leq n$, $1 \leq l < k$, then

$$(7.9) \quad (\mathcal{F}_k(g))^{\frac{1}{k}} \leq \binom{n}{k}^{\frac{1}{k}} \binom{n}{l}^{-\frac{1}{l}} (\mathcal{F}_l(g))^{\frac{1}{l}}.$$

(C). Moser-Trudinger type inequality: If $k = n/2$, then

$$(7.10) \quad (n - 2l)\mathcal{E}_{n/2}(g) \geq C_{MT} \left\{ \log \int_M \sigma_l(g^{-1}A_g) dV_g - \log \int_M \sigma_l(g_0^{-1}A_{g_0}) dV_{g_0} \right\},$$

Furthermore, the constants are all explicit and optimal, with a complete characterization of the case of equality see [GW04].

In the non-locally conformally flat case, recall from Theorem 3.1 that the σ_2 equation is always variational. Using this fact, the σ_2 -Yamabe equation has recently been studied in the general case using parabolic methods. In [GW05a], convergence was proved in dimensions $n > 8$, by constructing an explicit test function. In [STW05], convergence was proved in dimensions $n > 4$, who avoided having to directly construct a test function by employing the solution of the Yamabe problem. Combining this with the work described in Section 8, it follows that the σ_2 -Yamabe problem has been solved in all dimensions. Subsequently, the quotient equation σ_2/σ_1 was studied in [GW06] and existence was proved in dimensions $n > 4$.

8. POSITIVE RICCI CURVATURE

The ellipticity assumption of k -admissibility has geometric consequences on the Ricci curvature. The following inequality was demonstrated in [GVW03]:

Theorem 8.1. (*Guan-Viaclovsky-Wang [GVW03]*) Let (M, g) be a Riemannian manifold and $x \in M$. If $A_g \in \Gamma_k^+$ at x for some $k \geq n/2$, then its Ricci curvature is positive at x . Moreover, if $A_g \in \bar{\Gamma}_k^+$ for some $k > 1$, then

$$Ric_g \geq \frac{2k - n}{2n(k - 1)} R_g \cdot g.$$

In particular if $k \geq \frac{n}{2}$,

$$Ric_g \geq \frac{(2k - n)(n - 1)}{(k - 1)} \binom{n}{k}^{-\frac{1}{k}} \sigma_k^{\frac{1}{k}}(A_g) \cdot g.$$

Therefore, in the case $k > n/2$, any strictly k -admissible metric necessarily has positive Ricci curvature. This fact led to the following definition in [GV04b]:

Definition 8.2. Let (M^n, g) be a compact n -dimensional Riemannian manifold. For $n/2 \leq k \leq n$ the k -maximal volume of $[g]$ is

$$(8.1) \quad \Lambda_k(M^n, [g]) = \sup\{\text{vol}(e^{-2u}g) \mid e^{-2u}g \in \Gamma_k^+(M^n) \text{ with } \sigma_k^{1/k}(g_u^{-1}A_u) \geq \sigma_k\}.$$

If $[g]$ does not admit a k -admissible metric, set $\Lambda_k(M^n, [g]) = +\infty$.

Consider the σ_k -Yamabe equation

$$(8.2) \quad \sigma_k^{1/k} \left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g + A_g \right) = C_{n,k} e^{-2u}.$$

Where $C_{n,k}$ is the corresponding $\sigma_k^{1/k}$ -curvature of the standard round metric on S^n . Using Bishop's inequality, it follows that the invariant Λ_k is non-trivial when $k > n/2$, and in analogy with the classical Yamabe problem, when the invariant is strictly less than the value obtained by the round metric on the sphere one obtains existence of solutions to (8.2):

Theorem 8.3. (Gursky-Viaclovsky [GV04b]) If $[g]$ admits a k -admissible metric with $k > n/2$, then there is a constant $C = C(n)$ such that $\Lambda_k(M^n, [g]) < C(n)$. If

$$(8.3) \quad \Lambda_k(M^n, [g]) < \text{vol}(S^n),$$

where $\text{vol}(S^n)$ denotes the volume of the round sphere, then $[g]$ admits a solution $g_u = e^{-2u}g$ of (8.2). Furthermore, the set of solutions of (8.2) is compact in the C^m -topology for any $m \geq 0$.

The compactness proof uses a bubbling argument, combined with the Liouville Theorem of Li-Li [LL03], and existence is obtained using a degree-theoretic argument.

In dimension three, we have the following estimate

Theorem 8.4. (Gursky-Viaclovsky [GV04b]) Let (M^3, g) be a closed Riemannian three-manifold, and assume $[g]$ admits a k -admissible metric with $k = 2$ or 3 . Let $\pi_1(M^3)$ denote the fundamental group of M^3 . Then

$$(8.4) \quad \Lambda_k(M^3, [g]) \leq \frac{\text{vol}(S^3)}{\|\pi_1(M^3)\|}.$$

The proof of this used an improvement of Bishop's volume comparison theorem in dimension three, due to Hugh Bray [Bra97]. Therefore, if the three-manifold is not simply-connected, the estimate (8.3) is automatically satisfied.

In dimension four, the optimal estimate of Λ_k follows from the sharp integral estimate for $\sigma_2(A)$ due to Gursky [Gur99]:

Theorem 8.5. (Gursky-Viaclovsky [GV04b]) Let (M^4, g) be a closed Riemannian four-manifold, and assume $[g]$ admits a k -admissible metric with $2 \leq k \leq 4$. Then

$$(8.5) \quad \Lambda_k(M^4, [g]) \leq \text{vol}(S^4).$$

Furthermore, equality holds in (8.5) if and only if (M^4, g) is conformally equivalent to the round sphere.

When $k = 2$ the existence was established previously in [CGY02a]. Combining this work with the four-dimensional solution of the Yamabe problem [Sch84], it follows that the σ_k -Yamabe problem is completely solved in dimension four.

We return to the case of general dimension n . Using similar techniques as in the proof of Theorem 8.3, Guan-Wang studied the minimum eigenvalue of the Ricci, and the p -Weitzenböck operator, and proved various existence theorems, see [GW05b]. Their problem encountered some additional technical difficulties since their symmetric function F is not smooth, and also required an application of some work of Caffarelli [Caf89], and some generalizations of this work [Wan92a], [Wan92b].

In recent work, an existence theorem for a much more general class of F was proved. In addition to the structure conditions (i) – (iii) imposed on F above in Section 5, suppose that we also have

(iv) $\Gamma \supset \Gamma_n^+$, and there exists a constant $\delta > 0$ such that any $\lambda = (\lambda_1, \dots, \lambda_n) \in \Gamma$ satisfies

$$(8.6) \quad \lambda_i > -\frac{(1-2\delta)}{(n-2)}(\lambda_1 + \dots + \lambda_n) \quad \forall 1 \leq i \leq n.$$

To explain the significance of (8.6), suppose the eigenvalues of the Schouten tensor A_g are in Γ at each point of M^n . Then (M^n, g) has positive Ricci curvature: in fact,

$$(8.7) \quad Ric_g - 2\delta\sigma_1(A_g)g \geq 0.$$

Define

$$(8.8) \quad A_u \equiv \nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g + A_g.$$

For F satisfying (i) – (iv), consider the equation

$$(8.9) \quad F(g^{-1}A_u) = f(x)e^{-2u},$$

where we assume $g^{-1}A_u \in \Gamma$ (i.e., u is Γ -admissible).

Theorem 8.6. (*Gursky-Viaclovsky* [GV04a]) *Suppose $F : \Gamma \rightarrow \mathbb{R}$ satisfies (i) – (iv). Let (M^n, g) be closed n -dimensional Riemannian manifold, and assume*

(i) *g is Γ -admissible, and*

(ii) *(M^n, g) is not conformally equivalent to the round n -dimensional sphere.*

Then given any smooth positive function $f \in C^\infty(M^n)$ there exists a solution $u \in C^\infty(M^n)$ of

$$F(g^{-1}A_u) = f(x)e^{-2u},$$

and the set of all such solutions is compact in the C^m -topology for any $m \geq 0$.

This theorem in particular completely solved the σ_k -Yamabe problem whenever $k > n/2$. The proof of this theorem involved an bubbling analysis, and the application of various Holder and integral estimates to determine the growth rate of solutions at isolated singular points. For other works analyzing the possible behaviour of solutions

at singularities, see [CHY05], [GV06], [Gon04b],[Gon04a], [Li05a], [Li05b], [TW05]. In addition, a remarkable Harnack inequality for k -admissible metrics, $k > n/2$, was demonstrated in [TW05].

Note that the second assumption in the above theorem is of course necessary, since the set of solutions of (8.9) on the round sphere with $f(x) = \text{constant}$ is non-compact, while for variable f there are obstructions to existence. In particular, there is a ‘‘Pohozaev identity’’ for solutions of (8.9) in the case of σ_k , which holds in the conformally flat case; see [Via00c].

Theorem 8.7. (*Viaclovsky [Via00c]*) *Let (M, g) be a closed locally conformally flat n -dimensional manifold. Then for any conformal Killing vector field X , and $1 \leq k < n$, we have*

$$(8.10) \quad \int_M X \cdot \sigma_k(A) \, d\text{vol}_M = 0.$$

For $k = 1$, this identity is well-known and holds without the locally conformally flat assumption [Sch88]. This identity yields non-trivial Kazdan-Warner-type obstructions to existence (see [KW74]) in the case (M^n, g) is conformally equivalent to (S^n, g_{round}) . We note that similar Pohozaev-type identities were recently studied in [Han06], [Del06].

It is an interesting problem to characterize the functions $f(x)$ which may arise as σ_k -curvature functions in the conformal class of the round sphere. An announcement of some work by Chang-Han-Yang in this direction for σ_2 in dimension four was made in [Han04]. For $k = 1$, this problem is quite famous and has been studied in great depth. We do not attempt to make a complete list of references for this problem, we mention only [CGY93], [CL01], [ES86], [Li95], [Li96].

9. ADMISSIBLE METRICS

A natural question is: when does a manifold admit globally a strictly k -admissible metric, that is, a metric g with $A_g \in \Gamma_k^+$? In Section 6, we have already discussed the beautiful result for σ_2 in dimension four by Chang-Gursky-Yang. In the locally conformally flat case, there are various topological restrictions. Guan-Lin-Wang showed

Theorem 9.1. (*Guan-Lin-Wang [GLW05]*) *Let (M^n, g) be a compact, locally conformally flat manifold with $\sigma_1(A) > 0$.*

(i) *If $A \in \overline{\Gamma}_k^+$ for some $2 \leq k < n/2$, then the q th Betti number $b_q = 0$ for*

$$(9.1) \quad \left\lfloor \frac{n+1}{2} \right\rfloor + 1 - k \leq q \leq n - \left(\left\lfloor \frac{n+1}{2} \right\rfloor + 1 - k \right).$$

(ii) *Suppose $A \in \Gamma_2^+$, then $b_q = 0$ for*

$$(9.2) \quad \left\lfloor \frac{n - \sqrt{n}}{2} \right\rfloor \leq q \leq \left\lfloor \frac{n + \sqrt{n}}{2} \right\rfloor.$$

If $A \in \overline{\Gamma}_2^+$, $p = \frac{n - \sqrt{n}}{2}$, and $b_p \neq 0$, then (M, g) is a quotient of $S^{n-p} \times H^p$.

(iii) If $k \geq \frac{n-\sqrt{n}}{2}$ and $A \in \Gamma_k^+$, then $b_q = 0$ for any $2 \leq q \leq n-2$. If $k = \frac{n-\sqrt{n}}{2}$, $A \in \bar{\Gamma}_k^+$, and $b_2 \neq 0$, then (M, g) is a quotient of $S^{n-2} \times H^2$.

The proof of this theorem involves a careful analysis of the curvature terms in the Weitzenböck formula for the Hodge Laplacian. For σ_2 , the following was shown

Theorem 9.2. (Chang-Hang-Yang [CHY05]) *Let (M^n, g) , $(n \geq 5)$ be a smooth locally conformally flat Riemannian manifold such that $\sigma_1(A) > 0$, and $\sigma_2(A) \geq 0$, then $\pi_1(M) = 0$ for any $2 \leq i \leq [n/2]+1$, and $H^j(M, \mathbb{R}) = 0$ for any $n/2-1 \leq j \leq n/2+1$.*

Briefly, the technique in [CHY05] is to use the positivity condition to estimate the Hausdorff dimension of the singular set, using the developing map, as in [SY88]. Subsequently, in her thesis, María del Mar González generalized this to prove the following.

Theorem 9.3. (González [Gon05]) *Let (M, g) be compact, locally conformally flat, and $A \in \Gamma_k^+$, $k < n/2$. Then for any $2 \leq i \leq [\frac{n}{2}] + k - 1$, the homotopy group $\pi_i(M) = \{0\}$, and the cohomology group $H^i(M, \mathbb{R}) = \{0\}$ for $\frac{n-2k}{2} + 1 \leq i \leq \frac{n+2k}{2} - 1$.*

A beautiful gluing theorem was proved in [GLW05]:

Theorem 9.4. (Guan-Lin-Wang [GLW05]) *Let $2 \leq k < n/2$, and let M_1^n and M_2^n be two compact manifolds with $A_1, A_2 \in \Gamma_k^+$. Then the connected sum $M_1 \# M_2$ also admits a metric $g_\#$ with $A \in \Gamma_k^+$. If in addition, M_1 and M_2 are locally conformally flat, then $g_\#$ can also be taken to be locally conformally flat.*

This result can be viewed as a generalization of the analogous result for positive scalar curvature [GL80] [SY79b]. This yields many new examples of manifolds admitting metrics with $A \in \Gamma_k^+$, we refer the reader to [GLW05] for more details.

10. THE NEGATIVE CONE

All of the previous results are concerned with the positive curvature case. The negative curvature case exhibits quite different behaviour. In this case, a serious technical difficulty arises in attempting to derive *a priori* second derivative estimates on solutions [Via02]. Consider instead the following generalization of the Schouten tensor. Let $t \in \mathbb{R}$, and define

$$(10.1) \quad A^t = \frac{1}{n-2} \left(Ric - \frac{t}{2(n-1)} Rg \right).$$

We let $\Gamma_k^- = -\Gamma_k^+$.

Theorem 10.1. (Gursky-Viaclovsky [GV03b]) *Let (M, g) be a compact Riemannian manifold, assume that $A_g^t \in \Gamma_k^-$ for some $t < 1$, and let $f(x) < 0$ be any smooth function on M^n . Then there exists a unique conformal metric $\tilde{g} = e^{2w}g$ satisfying*

$$(10.2) \quad \sigma_k^{1/k}(\tilde{g}^{-1}A_{\tilde{g}}^t) = f(x).$$

As noted above, the second derivative estimate encounters technical difficulties for $t = 1$, but for $t < 1$, this difficulty can be overcome. Also, for $t = 1 + \epsilon$, the equation is not necessarily elliptic, therefore $t = 1$ is critical for more than one reason. Some local counterexamples to the second derivative estimate in the negative case have been given in [STW05], but there are no known global counterexamples. The above theorem was subsequently proved by parabolic methods in [LS05].

The above theorem has the following corollary. Using results of [Bro89], [GY86], and [Loh94], every compact manifold of dimension $n \geq 3$ admits a metric with negative Ricci curvature. Therefore applying the theorem when $t = 0$,

Corollary 10.2. (*Gursky-Viaclovsky* [GV03b]) *Every smooth compact n -manifold, $n \geq 3$, admits a Riemannian metric with $\text{Ric} < 0$ and*

$$(10.3) \quad \det(g^{-1}\text{Ric}) = \text{constant}.$$

It turns out the equation is also elliptic for $t \geq n - 1$, and this has some interesting consequences, see [SZ05] for details. We also mention that Mazzeo and Pacard considered the σ_k -Yamabe equation in the context of conformally compact metrics, and showed that the deformation problem is unobstructed [MP03].

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